Math 592: Descriptive Set Theory
Lecture 1

Part 1: Polish spaces.

1. Definitions and examples.

Def. A **Polish space** is a topological space which is completely metrizable and 2^{nd} ctbl (equiv, separable).

We define it as a top. space and not a metric bec. we don't care about a particular metric, but only about its existence. Also, our maps will be continuous and preserve any metric. We may choose a metric that's convenient to us, e.g.:

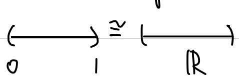
Obs (truncating the metric). If d is a complete metric compatible with the topology of a space X, then so is $d_{\leq 1}(x,y) := \min(d(x,y), 1)$.

Examples. (a) For all $n \in \mathbb{N}$, $n = \{0, 1, \dots, n-1\}$, n is Polish with the discrete topology. More generally, any ctbl set, like $\mathbb{N}$, equipped with the discrete top. is Polish.

(b) $\mathbb{R}$, $\mathbb{C}$, in fact, $\mathbb{R}^n$, $\mathbb{C}^n$, $n \geq 1$, are Polish.

(c) Any separable Banach, e.g. $\ell^p(\mathbb{N})$ or $L^p(\mathbb{R}, \lambda)$, $1 \leq p < \infty$, are Polish. In particular, any separable Hilbert space, $\ell^2(\mathbb{N})$ or $L^2(\mathbb{R}, \lambda)$, are Polish.

(d) Closed subsets of Polish space are Polish (witnessed by the same complete metric as for the whole space).

(e) $(0,1)$ 

is Polish bec. it is homeomorphic to $\mathbb{R}$.

Prop (closure properties).

(a) Completion of any separable metric space is Polish.

(b) Ctbl disjoint unions of Polish spaces are Polish.

(c) Ctbl products (with the product topology) of Polish spaces are Polish.

Proof. We only sketch (c), the rest is exercise. Let $(X_n)_{n\in\mathbb{N}}$ be sequence of Polish spaces and let d_n denote a compatible complete metric for X_n with $d_n \leq 1$. For $X := \prod_{n\in\mathbb{N}} X_n$, define, for $x,y \in X$,
$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n}\cdot d_n(x_n, y_n).$$
It is an exercise to show that d is a complete compatible metric for X. □

Examples. (a) $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$ are Polish.

(b) The Hilbert cube $[0,1]^{\mathbb{N}}$ is Polish (and compact).

(c) The Cantor space $2^{\mathbb{N}}$ is Polish (and compact), where $2 := \{0,1\}$ is with the discr. top.

(d) The Baire space $\mathbb{N}^{\mathbb{N}}$ is Polish (very non-compact), where $\mathbb{N}$ is with the discrete top.

Prop (binary representation). Every Polish space X (in fact, 2nd ctbl Hausdorff suffices) admits an injection $b : X \hookrightarrow 2^{\mathbb{N}}$ into the Cantor space such that the b-preimages of the clopen sets $[n\mapsto 1] := \{x \in 2^{\mathbb{N}} : x(n) = 1\}$ are open, i.e. $b^{-1}[n\mapsto 1]$ is open.

Proof. Fix a ctbl open basis $(U_n)_{n\in\mathbb{N}}$ for X. Define $b : X \to 2^{\mathbb{N}}$ by recording for each x whether it is in U_n or not: $x \mapsto b(x)$, where $b(x)(n) := \begin{cases} 1 & \text{if } x \in U_n \\ 0 & \text{otherwise.}\end{cases}$

This map is injective by Hausdorffness and by definition, $b^{-1}[n\mapsto 1] = U_n$. □

Cor. Polish spaces are at most of cardinality continuum $(= |\mathbb{R}| = |2^{\mathbb{N}}|)$.

Lemma. If X is metric space, then closed sets are G_δ (ctbl intersection of open sets); open sets are F_σ (ctbl unions of closed sets).

Proof. For any set $Y \subseteq X$, $\overline{Y} = \bigcap_{n\geq 1} B_{1/n}(Y)$, where $B_r(Y) := \{x \in X : d(x,Y) < r\} = \bigcup_{y\in Y} B_r(y)$. Thus, if Y is closed, then $Y = \bigcap_{n\geq 1} B_{1/n}(Y)$ and $B_{1/n}(Y)$ are open. □

Theorem. A subset Y of a Polish space X is Polish (in the subspace topology) $\Leftrightarrow$ Y is G_δ. In particular, open and closed subsets of Polish spaces are Polish.

Proof. $\Leftarrow$. Fix a complete compatible metric d for X. Let's first prove for an open set $U \subseteq X$. Like with $(0,1)$, we will show that

U is homeomorphic to a Polish space, specifically, a closed subset of $X \times \mathbb{R}$.
Take $f: U \to X \times \mathbb{R}$ by $x \mapsto (x, 1/d(x, U^c))$. $f(U)$ is closed because if $(x_n, 1/d(x_n, U^c)) \in f(U)$ and $\lim_n (x_n, 1/d(x_n, U^c)) = (x, r)$, then because r is finite, x must be in U.
One has to still verify that f is continuous (obvious) and $f^{-1}: f(U) \to X$ is continuous (also straightforward).

Now for a G_δ set $Y := \bigcap_{n \in \mathbb{N}} U_n$, U_n open, we do the same but for all the U_n:
$f: Y \to X \times \mathbb{R}^{\mathbb{N}}$ by $y \mapsto (y, (1/d(y, U_n^c))_{n \in \mathbb{N}})$. This is again an embedding with a closed image.

$\Rightarrow$ (Alexandrov). Let $Y \subseteq X$ be a Polish space itself, let d_X, d_Y be complete compatible metrics for X and Y, respectively. For each $n \in \mathbb{N}$, define an open set V_n as the union of all open sets $U \subseteq X$ satisfying the following:

$$\left.\begin{array}{l}\text{(i) } U \cap Y \neq \emptyset \\ \text{(ii) } \operatorname{diam}_{d_X}(U) < \frac{1}{n}\end{array}\right\} \Rightarrow V_n \subseteq B^X_{1/n}(Y).$$

(iii) $\operatorname{diam}_{d_Y}(U \cap Y) < \frac{1}{n}$.